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DATE 03-09-2020

Name of college — S. S. College, J-Bad
Deptt — Mathematics
Topic — Integration of Some special Trigonometric Functions.
Time — 10.15 A.M To 11.00 A.M
11.00 A.M To 11.45 A.M
Date — 03-09-2020
By — Dr. Shree Nath Sharma.

Study Material →

Transcendental Function →

Such a Functions which are not algebraic,

Ex → Trigonometric Functions, Exponential functions, Logarithmic Functions, hyperbolic Functi are Transcendental functions.

1. Form :→ $\int \sin^n x \, dx$ or $\int \cos^n x \, dx$

If n is an odd positive integer
put $\cos x = z$ and $\sin x = z$ respectivel

If n is small even +ve integer, the integrals can be carried out by Expressing the integrand in a series of sines cosines of multiple angles. If n is Larg use De Moiver's theorem.

Integrate

Ex:→ 1 $\int \sin^5 x \, dx$ (ii) $\int \cos^7 x \, dx$

(i) $I = \int \sin^5 x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \, dx$

put $\cos x = z$

$\sin x \, dx = -dz$

$= -\int (1-z^2)^2 dz$

$= -\int (1+z^4-2z^2) dz$

$= -\left[z + \frac{z^5}{5} - \frac{2z^3}{3} \right] + C$

$= -\left[\cos x + \frac{\cos^5 x}{5} - \frac{2\cos^3 x}{3} \right] + C$

(ii) $I = \int \cos^7 x \, dx$

$= \int (1-\sin^2 x)^3 \cos x \, dx$

$\sin x = z$

$\Rightarrow \cos x \, dx = dz$

$= \int (1-z^2)^3 dz$

$= \int \{1 - 3z^2 + 3z^4 - z^6\} dz$

$= z - \frac{3z^3}{3} + \frac{3z^5}{5} - \frac{z^7}{7} + C$

$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{\sin^7 x}{7} + C$

(iii) $I = \int \sin^6 x \, dx$

(iv) $\int \cos^8 x \, dx$

put $\cos x + i\sin x = z$

$\cos x - i\sin x = \frac{1}{z}$

$2i\sin x = z - \frac{1}{z}$

$(2i\sin x)^6 = (z-\frac{1}{z})^6$

$= {}^6C_0 z^6 - {}^6C_1 z^5 \cdot \frac{1}{z} + {}^6C_2 z^4 \cdot \frac{1}{z^2} - {}^6C_3 z^3 \frac{1}{z^3}$

$+ {}^6C_4 z^2 \cdot \frac{1}{z^4} - {}^6C_5 z \cdot \frac{1}{z^5} + {}^6C_6 \cdot \frac{1}{z^6}$

$= z^6 - 6z^4 + 15z^2 - 20 + 15 \cdot \frac{1}{z^2} - 6\frac{1}{z^4} \cdot$

$\cancel{\left(z^6+\frac{1}{z^6}\right) - 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) - 20}$

$\therefore 2^6 i^6 \sin^6 x = 2[\cos 6x - 6\cos 4x + 15\cos 2x]$

$\Rightarrow -2^6 \sin^6 x = 2\cos 6x - 12\cos 4x + 30\cos 2x - 2$

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$$\Rightarrow -2^6 \sin^6 x = 2\left[\cos 6x - 6\cos 4x + 15\cos 2x - 10\right]$$

$$\sin^6 x = -\frac{1}{2^5}\left[\cos 6x - 6\cos 4x + 15\cos 2x - 10\right]$$

$$\therefore \int \sin^6 x\,dx = -\frac{1}{32}\left[\frac{\sin 6x}{6} - \frac{6\sin 4x}{4} + \frac{15\sin 2x}{2} - 10x\right] + C$$

Let $\cos x + i\sin x = z \Rightarrow z^n = \cos nx + i\sin nx$

$\cos x - i\sin x = \frac{1}{z}$ $\quad \frac{1}{z^n} = \cos nx - i\sin nx$

$z^n + \frac{1}{z^n} = 2\cos nx \quad z^n - \frac{1}{z^n} = 2i\sin nx$

$\Rightarrow 2\cos x = z + \frac{1}{z}$

$$\Rightarrow 2^8\cos^8 x = \left[z + \frac{1}{z}\right]^8 = z^8 + 8C_1 z^7\cdot\frac{1}{z} + 8C_2 z^6\cdot\frac{1}{z^2}$$
$$+ 8C_3 z^5\cdot\frac{1}{z^3} + 8C_4 z^4\frac{1}{z^4} + 8C_5 z^3\frac{1}{z^5}$$
$$+ 8C_6 z^2\cdot\frac{1}{z^6} + 8C_7 z\cdot\frac{1}{z^7} + \frac{1}{z^8}$$

$$\Rightarrow 2^8\cos^8 x = z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + 56\frac{1}{z^2}$$
$$+ 28\cdot\frac{1}{z^4} + 8\frac{1}{z^6} + \frac{1}{z^8}$$
$$= \left(z^8 + \frac{1}{z^8}\right) + 8\left(z^6 + \frac{1}{z^6}\right) + 28\left(z^4 + \frac{1}{z^4}\right) + 56\left(z^2 + \frac{1}{z^2}\right)$$
$$+ 70$$
$$= 2\left[\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x\right] + 70$$
$$= 2\left[\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35\right]$$

$$\Rightarrow \cos^8 x = \frac{1}{2^7}\left(\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35\right)$$

$$\therefore \int \cos^8 x\,dx = \frac{1}{2^7}\left[\frac{\sin 8x}{8} + 8\frac{\sin 6x}{6} + \frac{28\sin 4x}{4} + \frac{56\sin 2x}{2}\right. + C$$

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$$\therefore \int \cos^8 x\,dx = \frac{1}{128}\left[\frac{\sin 8x}{8} + \frac{8\sin 6x}{6} + \frac{28\sin 4x}{4} + \frac{56\sin 2x}{2} + 35x\right] + c$$

Form 2. $\int \tan^n x\,dx$

Ex:→ $\int \tan^8 x\,dx$

$$= \int \tan^6 x(\sec^2 x - 1)\,dx$$

$$= \int \tan^6 x \sec^2 x\,dx - \int \tan^4 x(\sec^2 x - 1)\,dx$$

$$= \int \tan^6 x \sec^2 x\,dx - \int \tan^4 x \sec^2 x\,dx + \int \tan^2 x(\sec^2 x - 1)\,dx$$

$$= \int \tan^6 x \sec^2 x\,dx - \int \tan^4 x \sec^2 x + \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$

Put $\tan x = z$
$\sec^2 x\,dx = dz$

$$= \int z^6\,dz - \int z^4\,dz + \int z^2\,dx - \int \sec^2 x\,dx + \int dx$$

$$= \frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - \tan x + x + c$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x + C$$

Form - $\int \sin^m x \cos^n x\,dx$

sin x = z if n is odd

put either sin x = z or cos x = z if m, n both are odd

put tan x = z if m + n = -ve even integer

Integrate

(1) $\int \frac{\sin^5 x}{\sqrt{\cos x}}\, dx$

Integral $= \int \frac{\sin^5 x}{\sqrt{\cos x}}\, dx$

$= \int (\cos x)^{-1/2} \sin^5 x \, dx$

$= \int (\cos x)^{-1/2} (1-\cos^2 x)^2 \sin x \, dx$

$= \int (\cos x)^{-1/2} [1 - 2\cos^2 x + \cos^4 x] \sin x \, dx$

put $\cos x = z$

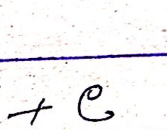
$\sin x \, dx = -dz$

$= -\int z^{-1/2} (1 - 2z^2 + z^4)\, dz$

$= -\left[\int z^{-1/2} dz - 2\int z^{3/2} dz + \int z^{7/2} dz\right]$

$= -\left[2z^{1/2} - \frac{2 \times 2z^{5/2}}{5} + \frac{2z^{9/2}}{9}\right] + C$

$= -2\sqrt{\cos x} + \frac{4}{5}\cos^{5/2} x + \frac{2}{9}\cos^{9/2} x + C$



(ii) $I = \int \frac{dx}{\sin^4 x \cos^2 x}\, dx$

put $\tan x = z \Rightarrow \sin x = \frac{z}{\sqrt{1+z^2}}$

$\Rightarrow \sec^2 x \, dx = dz$

$\cos x = \frac{1}{\sqrt{1+z^2}}$

$$= \int \frac{\sec^2 x \, dx}{\sin^4 x}$$

$$= \int \frac{(1+z^2)^2 dz}{z^4}$$

$$= \int \frac{1+2z^2+z^4}{z^4} dz$$

$$= \int z^{-4} dz + 2\int z^{-2} dz + \int dz$$

$$= -\frac{1}{z^3} - \frac{2}{z} + z + C$$

$$= -\frac{1}{\tan^3 x} - \frac{2}{\tan x} + \tan x + C$$

$$= -\cot^3 x - 2\cot x + \tan x + C$$

1.

Form $I = \int \frac{1}{a+b\cos x} dx$

$$a+b\cos x = a+b \cdot \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

$$= \frac{1}{1+\tan^2 x/2}\left[a(1+\tan^2 \tfrac{x}{2}) + b(1-\tan^2 x/2)\right]$$

$$= \frac{1}{\sec^2 \frac{x}{2}}\left[(a+b) + (a-b)\tan^2 \tfrac{x}{2}\right]$$

$$= \frac{a-b}{\sec^2 x/2}\left[\frac{a+b}{a-b} + \tan^2 x/2\right]$$

$$I = \frac{1}{a-b}\int \frac{\sec^2 x/2}{\frac{a+b}{a-b} + \tan^2 x/2}$$

put $\tan x/2 = z$

$\frac{1}{2}\sec^2 x/2 \, dx = dz$

$\sec^2 x/2 \, dx = 2dz$

$\therefore I = \frac{2}{a-b}\int \frac{dz}{\frac{a+b}{a-b}+z^2}$

case if $a>b$

$I = \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left[z\sqrt{\frac{a-b}{a+b}}\right]$ $\quad \because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

If $a<b$.

$= \frac{2}{b-a}\int\frac{dz}{\frac{b+a}{b-a}-z^2}$

$= \frac{2}{b-a}\cdot\frac{\sqrt{b-a}}{2\sqrt{b+a}}\log\frac{\sqrt{\frac{b+a}{b-a}}+z}{\sqrt{\frac{b+a}{b-a}}-z}$ $\quad \because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

Form $\int\frac{dx}{a+b\sin x}$

$= \int\frac{dx}{a\cos^2\frac{x}{2}+a\sin^2\frac{x}{2}+2b\sin x/2\cos x/2}$

$= \int\frac{dx}{\cos^2 x/2\,[a+a\tan^2 x/2+2b\tan x/2]}$

$= \int\frac{\sec^2\frac{x}{2}\,dx}{a+a\tan^2 x/2+2b\tan x/2}$ $\quad$ put $\tan x/2 = z$

$\sec^2 x/2\,dx = 2\,dz$

$= 2\int\frac{\cancel{2}\,dz}{az^2+2bz+a}$

$= \frac{2}{a}\int\frac{\cancel{2}\,dz}{z^2+\frac{2bz}{a}+\cancel{a}1}$ $\quad = \frac{2}{a}\int\frac{dz}{z^2+\frac{2b}{a}z+1}$

$$= \frac{2}{a}\int \frac{dz}{(z+\frac{b}{a})^2 + 1 - \frac{b^2}{a^2}}$$

$$= \frac{2}{a}\int \frac{dz}{(z+\frac{b}{a})^2 - \left(\frac{\sqrt{a^2-b^2}}{a}\right)^2}$$

When $a^2 > b^2$

$$\int \frac{dx}{a+b\sin x} = \frac{2}{a}\cdot\frac{a}{\sqrt{a^2-b^2}} \tan^{-1}\left\{(z+\frac{b}{a})\cdot\frac{a}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left[\frac{az+b}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a\tan x/2 + b}{\sqrt{a^2-b^2}}$$

Case II - If $a^2 < b^2$ then

$$\int \frac{dx}{a+b\sin x} = \frac{2}{a}\int \frac{dz}{(z+b/a)^2 - \left(\frac{\sqrt{b^2-a^2}}{a}\right)^2}$$

$$= \frac{2}{a}\cdot\frac{1}{2}\frac{a}{\sqrt{b^2-a^2}} \log \frac{z+b/a - \frac{\sqrt{b^2-a^2}}{a}}{z+\frac{b}{a} + \frac{\sqrt{b^2-a^2}}{a}}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{az+b-\sqrt{b^2-a^2}}{az+b+\sqrt{b^2-a^2}}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{a\tan x/2 + b - \sqrt{b^2-a^2}}{a\tan x/2 + b + \sqrt{b^2-a^2}}$$